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Bornological spaces of non-Archimedean valued functions

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ABSTRACT

Let $C(X,E)$ be the space of all continuous functions from an ultraregular space X to a non-Archimedean locally convex space E. Necessary and/or sufficient conditions are given so that $C(X,E)$, with the topology of uniform convergence on compact sets or with the topology of simple convergence, is bornological or c-ultrabornological.
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INTRODUCTION

Let X be an ultraregular space, K a complete non-Archimedean non-trivially valued field, and $C(X,K)$ the space of all continuous K-valued functions on X. Govaerts has shown in [2] and [3] that $C(X,K)$, with the topology of simple convergence or with the topology of compact convergence, is bornological iff X is N-replete. In this paper, we show that the same is true if we replace K with an arbitrary metrizable locally K-convex space E. If K is spherically complete and X satisfies the first axiom of countability, then for an arbitrary locally K-convex space E, the space $C(X,E)$, with the topology of simple convergence or with the topology of compact convergence, is bornological iff X is N-replete and E is bornological. It is also shown that, if E is a Fréchet space, then $C(X,E)$, with the topology of compact convergence, is c-ultrabornological iff X is N-replete and K is spherically complete.

1. PRELIMINARIES

Throughout this paper, X will be an ultraregular topological space, i.e. X is a Hausdorff space for which the clopen sets form a base for the topology. The Banaschewski compactification $\beta_0 X$ of X (see [1]) is a compact ultraregular space that contains X as a dense subspace and is such that disjoint clopen subsets of X have closures in $\beta_0 X$ which are clopen and disjoint. As in [1], $\upsilon_0 X$ is the set of all $x \in \beta_0 X$ such that for each sequence (V_n) of neighborhoods of x in $\beta_0 X$ we have $\bigcap_{n=1}^{\infty} V_n \cap X \neq \emptyset$. By [1, Theorem 9], $\upsilon_0 X$ coincides with the N-repletion $\upsilon_N X$ of X, where N is the set of all positive integers. Let K be a complete non-Archimedean non-trivially valued field, E a Hausdorff locally K-convex space, and $C(X,E)$ the space of all continuous E-valued functions on X. We will denote by $C_s(X,E)$ (respectively $C_c(X,E)$) the space $C(X,E)$ equipped with the topology t_s of simple convergence (resp. with the topology t_c of compact convergence). For a subset A of X, we will denote by χ_A the K-characteristic function of A. Also, if f is a function from X to E and p a seminorm on E, we define

$$\|f\|_{A,p} = \sup \{p(f(x)) : x \in A\}, \quad \|f\|_p = \|f\|_{X,p}.$$

For a function g, from X to a non-Archimedean normed space G, we define

$$\|f\|_A = \sup \{\|g(x)\| : x \in A\}, \quad \|g\| = \|g\|_X.$$

Let p be a continuous non-Archimedean seminorm on E and set $G_p = \{p(s) : s \in E\}$. On G_p we consider the ultrametric d defined by

$$\begin{aligned} d(x,y) &= 0 && \text{if } x = y, \\ &= \max(x,y) && \text{if } x \neq y. \end{aligned}$$

Under this metric, G_p becomes a realcompact, strongly ultraregular, non-compact topological space and so $\upsilon_{G_p} X = \upsilon_0 X = \upsilon_N X$ (see [1, Theorem 9]). If f is in $C(X,E)$, then the function $f_p : X \to G_p$, $f_p(x) = p(f(x))$, is continuous and so it has a continuous extension $\tilde{f}_p$ to all of $\upsilon_0 X$.

For a locally K-convex space F, we denote by F'_σ the space F' with the weak topology $\sigma(F,F')$. For K spherically complete, the finest locally K-convex topology on F, compatible with the pair $\langle F,F' \rangle$, will be denoted by $\tau_c(F,F')$ (see [11]). For the notion of c-compactness, we refer to [10].

A locally K-convex space F is called bornological if every absolutely convex subset of F absorbing bounded sets is a neighborhood of zero. The space F is called c-ultrabornological if every absolutely convex subset of F absorbing absolutely convex, bounded, and c-compact sets is a neighborhood of zero.

Let now $S(X)$ be the algebra of all clopen subsets of X. We will denote by $M(X,E')$ the space of all finitely additive E'-valued measures m on $S(X)$ for which $m(S(X))$ is an eqicontinuous subset of E' (see [9]). Let $A \in S(X)$, $A \neq \emptyset$. Consider the family Φ_A of all $\alpha = \{A_1, \ldots, A_n; x_1, \ldots, x_n\}$ where $A_1, \ldots, A_n$ is a clopen partition of A and $x_i \in A_i$. The set Φ_A becomes directed by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the partition in α_2. If f is an E-valued function on X and $\alpha = \{A_1, \ldots, A_n; x_1, \ldots, x_n\}$ in Φ_A, we define

$$\omega_\alpha(f,m) = \sum_{i=1}^{n} m(A_i) f(x_i).$$

If the limit $\lim_{\alpha \in \Phi_A} \omega_\alpha(f, m)$ exists, then f is said to be integrable over A and we denote this limit by $\int_A f dm$. We write simply $\int f dm$ for $\int_X f dm$. For $\phi \in E'$ and $x \in X$, we denote by $m_{x,\phi}$ the element of $M(X, E')$ defined by $m_{x,\phi}(A) = \chi_A(x)\phi$. We have $m_{x,\phi}(f) = \phi(f(x))$ for all $f \in C(X, E)$.

Finally, the complement of a subset A of X will be denoted by A^c or by $X \setminus A$.

2. BORNOLOGICAL SPACES $C(X, E)$

THEOREM 2.1. If $C_c(X, E)$ or $C_s(X, E)$ is bornological, then E is bornological and X is N-replete.

PROOF. The proof for $C_s(X, E)$ is given in [8, theorem 3.1]. For the case of $C_c(X, E)$ we can use an analogous argument.

NOTATION. For a non-empty absolutely convex subset D of $C(X, E)$, we will let Ω_D denote the family of all clopen subsets A of X which have the following property: If $f \in C(X, E)$ vanishes on A, then $f \in D$. The intersection

$$\bigcap \{\bar{A}^{\beta_0 X} : A \in \Omega_D\}$$

will be denoted by $F(D)$.

PROPOSITION 2.2. (a) If A_1, A_2 are in Ω_D, then $A = A_1 \cap A_2 \in \Omega_D$.

(b) If A is a clopen subset of X such that $\bar{A}^{\beta_0 X} \supset F(D)$, then $A \in \Omega_D$.

(c) An element $x \in \beta_0 X$ belongs to $F(D)$ iff the following holds: For each clopen neighborhood W of x in $\beta_0 X$ there exists $f \in C(X, E)$ with $f = 0$ on $X \setminus W \cap X$ and $f \notin D$.

(d) $F(D)$ is the smallest compact subset G of $\beta_0 X$ with the following property: If $f \in C(X, E)$ vanishes on a clopen subset W of X with $\bar{W}^{\beta_0 X} \supset G$, then $f \in D$.

PROOF. (a) Let $f \in C(X, E)$ vanishing on A. If $B_1 = A_1^c$ and $B_2 = A_1 \cap A_2^c$, then $f = f \cdot \chi_{B_1} + f \cdot \chi_{B_2} \in D$.

(b) Let A be a clopen subset of X whose closure in $\beta_0 X$ contains $F(D)$. The set $\bar{A}^{\beta_0 X}$ is clopen and so its complement G in $\beta_0 X$ is compact. Hence there are $B_1, \ldots, B_n \in \Omega_D$ such that G is covered by the complements of the sets $\bar{B}_i^{\beta_0 X}$, $i = 1, \ldots, n$. Hence $G \subset \beta_0 X \setminus \bar{B}^{\beta_0 X}$, $B = \bigcap_{i=1}^n B_i \in \Omega_D$ and so

$$B = \bar{B}^{\beta_0 X} \cap X \subset \bar{A}^{\beta_0 X} \cap X = A$$

which implies that $A \in \Omega_D$.

(c) ($\Rightarrow$) The set $A = X \cap [\beta_0 X \setminus W] = X \setminus W \cap X$ is clopen in X with $\bar{A}^{\beta_0 X} = \beta_0 X \setminus W$ from which it follows that $A \notin \Omega_D$ and hence the claim is clear. ($\Leftarrow$). If $x \notin F(D)$, then there exists $A \in \Omega_D$ with $x \notin \bar{A}^{\beta_0 X}$. The set $W = \beta_0 X \setminus \bar{A}^{\beta_0 X}$ is clopen, contains x and $X \setminus W \cap X = X \cap \bar{A}^{\beta_0 X} = A$. Hence every f which vanishes on $X \setminus W \cap X$ belongs to D.

(d) Let G be a compact subset of $\beta_0 X$ for which the property is satisfied.

There exists a family $\{W_\alpha\}$ of clopen sets in $\beta_0 X$ with $G=\bigcap W_\alpha$. If $V_\alpha = W_\alpha \cap X$, then W_α is the closure of V_α in $\beta_0 X$. Our hypothesis implies that each V_α is in Ω_D from which it follows that $G \supset F(D)$. This, together with (b), complete the proof.

PROPOSITION 2.3. Suppose that D absorbs t_c-bounded sets. Then:

(a) If (A_n) is a sequence of pairwise disjoint clopen sets in X such that $A=\bigcup A_n$ is clopen, then there exists n_0 such that $X \setminus A_n \in \Omega_D$ if $n \geq n_0$.

(b) If (V_n) is a sequence in Ω_D such that $V=\bigcap V_n$ is clopen, then $V \in \Omega_D$.

PROOF. (a) Assume the contrary and let $N_1=\{n \in N : X \setminus A_n \notin \Omega_D\}$, $N_2 = N \setminus N_1$. The set $B_1 = \bigcup_{n \in N_1} A_n$ is open. Also $X \setminus B_1 = X \setminus A \cup [\bigcup_{n \in N_2} A_n]$ is open. Thus B_1 is clopen and so we may assume that $X \setminus A_n \notin \Omega_D$ for each n. Let f_n in $C(X,E)$ with $f_n = 0$ on $X \setminus A_n$ and $f_n \notin D$. If $|\lambda|>1$, then the set $B=\{\lambda^n f_n : n \in N\}$ is t_c-bounded. In fact, if F is a compact subset of X, then there exists m such that $F \cap A \subset \bigcup_{n \leq m} A_n$. Hence, for $k>m$, we have $f_k=0$ on F. Let now $\mu \in K$ with $B \subset \mu D$. But then, for $|\lambda^n| \geq |\mu|$, we have $f_n \in D$ which is a contradiction.

(b) By Proposition 2.2, we may assume that (V_n) is decreasing. Let now f in $C(X,E)$ vanishing on V and let $|\lambda|>1$. The set $G=\{\lambda^n f \chi_{V_n} : n \in N\}$ is t_c-bounded. In fact, if F is compact in X, then there exists n such that $F \cap V^c \subset V_n^c$. If now $m>n$, then $f\chi_{V_m}$ vanishes on F which proves that G is t_c-bounded. It follows that $f\chi_{V_m} \in D$ for some m. Since $f\chi_{V_m^c} \in D$, we have $f = f\chi_{V_m} + f\chi_{V_m^c} \in D$ and this completes the proof.

PROPOSITION 2.4 If D absorbs t_c-bounded sets, then $F(D) \subset \upsilon_0 X$.

PROOF. Suppose that $x \in F(D)$ with $x \notin \upsilon_0 X$. Then, there exists a decreasing sequence (A_n) of clopen neighborhoods of x in $\beta_0 X$ such that $A_1 = \beta_0 X$ and $\bigcap_{n=1} A_n \cap X = \emptyset$. Set $G_n = (A_n \setminus A_{n+1}) \cap X$. Then $\bigcup_n G_n = X$. By the preceding Proposition, there exists m such that $X \setminus G_n \in \Omega_D$ if $n \geq m$. Since

$$\bigcap_{n \geq m} (X \setminus G_n) = X \setminus A_m \cap X$$

is clopen, we have $X \setminus A_m \cap X \in \Omega_D$ by the preceding Proposition, and so $x \in \overline{X \setminus A_m \cap X}^{\beta_0 X}$. On the other hand, $x \in \overline{A_m \cap X}^{\beta_0 X} = A_m$. This is a contradiction since the disjoint clopen sets $A_m \cap X$ and $X \setminus A_m \cap X$ must have disjoint closures in $\beta_0 X$ (see [1, Theorem 5]).

LEMMA 2.5. Let (x_n) be a sequence of distinct elements in an ultraregular space Y. Then, there exists a subsequence (x_{n_k}) and pairwise disjoint clopen sets V_k with $x_{n_k} \in V_k$.

PROOF. If $x_n \to x_1$, take $n_1 = 2$, otherwise take $n_1 = 1$. Since $x_n \not\to x_{n_1}$, there exists a clopen neighborhood V_1 of x_{n_1} and an infinite subset N_1 of N such that $x_n \notin V_1$ for each $n \in N_1$. Let $n_2 > n_1$ such that the sequence $(x_n)_{n \in N_1}$ does not converge to x_{n_2}. There exists a clopen neighborhood $V_2 \subset V_1^c$ of x_{n_2} and an infinite subset N_2 of N_1 such that $x_n \notin V_2$ if $n \in N_2$. Continuing in this way we get by induction our result.

PROPOSITION 2.6. If D absorbs t_s-bounded sets, then $F(D)$ is a finite subset of $\upsilon_0 X$.

PROOF. By Proposition 2.4, $F(D) \subset \upsilon_0 X$. Suppose that $F(D)$ is infinite. By the preceding Lemma, there exist a sequence (x_n) in $F(D)$ and a sequence (V_n) of pairwise disjoint clopen subsets of $\beta_0 X$ with $x_n \in V_n$. By Proposition 2.2 (c), there exists $f_n \in C(X,E)$, $f_n \notin D$, with $f_n = 0$ on $X \setminus V_n \cap X$. Let $|\lambda| > 1$. The set $B = \{\lambda^n f_n : n \in N\}$ is t_s-bounded. Hence, there exists λ_0 with $B \subset \lambda_0 D$ and so $f_n \in D$ if $|\lambda| \geq |\lambda_0|$. This contradiction completes the proof.

PROPOSITION 2.7. Suppose that $F(D) \subset \upsilon_0 X$ and that there exists a continuous non-Archimedean seminorm p on E and $r > 0$ such that

$$V_1 = \{f \in C(X,E) : \|f\|_p \leq r\} \subset D.$$

Then

$$V_2 = \{f \in C(X,E) : \tilde{f}_p(x) \leq r \text{ for all } x \in F(D)\} \subset D.$$

PROOF. Let $f \in V_2$. The set $W = \{x \in X : f_p(x) \leq r\}$ is clopen and its closure in $\beta_0 X$ contains $F(D)$. Thus, by Proposition 2.2, $W \in \Omega_D$ and so $f\chi_W^c \in D$. Also $f\chi_W \in V_1 \subset D$ and hence $f \in D$.

PROPOSITION 2.8. Let E be metrizable and let D be an absolutely convex subset of $C(X,E)$ absorbing t_c-bounded sets. Then, there exists a non-Archimedean continuous seminorm p on E and $r > 0$ such that

$$(*) \qquad \{f \in C(X,E) : \|f\|_p \leq r\} \subset D.$$

PROOF. Let (p_n) be an increasing sequence of non-Archimedean seminorms on E generating its topology and suppose that we cannot find p, r such that $(*)$ holds. If $|\lambda| > 1$, then, for each $n \in N$, there exists $f_n \in C(X,E)$ with $\|f_n\|_{p_n} \leq |\lambda|^{-n}$ and $f_n \notin D$. The set $B = \{\lambda^n f_n : n \in N\}$ is t_c-bounded and so $B \subset \lambda_0 D$ for some λ_0. But then $f_n \in D$ if $|\lambda|^n \geq |\lambda_0|$, which is a contradiction.

Combining Theorem 2.1 with Propositions 2.4, 2.6, 2.7 and 2.8, we get

THEOREM 2.9. If E is metrizable, then the following are equivalent:
(1) X is N-replete;
(2) $C_s(X,E)$ is bornological;
(3) $C_c(X,E)$ is bornological.

PROOF. If $C_s(X,E)$ or $C_c(X,E)$ is bornological, then X is N-replete by Theorem 2.1. Conversely, let X be N-replete. Then $\upsilon_0 X = X$. If D is an absolutely convex subset of $C(X,E)$ absorbing t_s (resp. t_c) bounded sets, then $F(D)$ is a finite (resp. compact) subset of $\upsilon_0 X = X$ by Propositions 2.6 and 2.5. By Proposition 2.8, there exist a non-Archimedean continuous seminorm p on E and $r>0$ such that

$$\{f \in C(X,E) : \|f\|_p \leq r\} \subset D,$$

and so

$$\{f \in C(X,E) : \|f\|_{F(D)} \leq r\} \subset D$$

by Proposition 2.7. Hence D is a t_s (resp. a t_c) neighborhood of zero and this completes the proof.

LEMMA 2.10. Let Y be an ultraregular space satisfying the first axiom of countability. Then, for each finite subset $S=\{x_1, \ldots, x_m\}$ of Y, there exists a decreasing sequence (W_n) of clopen sets with $\cap W_n = S$.

PROOF. For each k, $1 \leq k \leq m$, there exists a decreasing sequence $(V_{k,n})_{n=1}^{\infty}$ of clopen neighborhoods of x_k which is a base at x_k. Then $\bigcap_{n=1}^{\infty} V_{k,n} = \{x_k\}$. Now it suffices to take $W_n = \bigcup_{k=1}^{m} V_{k,n}$.

PROPOSITION 2.11. Suppose that $\upsilon_0 X$ satisfies the first axiom of countability and that the absolutely convex subset D of $C(X,E)$ absorbs t_s-bounded sets. Then, any $f \in C(X,E)$ vanishing on $F(D) \cap X$ belongs to D.

PROOF. By Proposition 2.6, $F(D)$ is a finite subset of $\upsilon_0 X$. Hence, by the preceding Lemma, there exists a decreasing sequence (W_n) of clopen sets in $\upsilon_0 X$ with $\cap W_n = F(D)$. Let $f \in C(X,E)$ vanish on $F(D) \cap X$. If $|\lambda| > 1$ and $V_n = W_n \cap X$, then the sequence $(\lambda^n \chi_{V_n} f)$ is t_s-bounded, from which it follows that there exists n such that $f_1 = \chi_{V_n} f \in D$. Also $f_2 = \chi_{V_n^c} \cdot f \in D$ since it vanishes on V_n and the closure of V_n in $\beta_0 X$ contains $W_n \supset F(D)$. Thus $f = f_1 + f_2 \in D$.

LEMMA 2.12. Let K be spherically complete and let F be a Hausdorff locally K-convex space. Then, every absolutely convex, weakly bounded and weakly c-compact subset of F' is strongly bounded.

PROOF. The proof is analogous to the one in the classical case.

LEMMA 2.13. Let K be spherically complete and let F be a Hausdorff locally K-convex space. Then F is bornological iff the following two conditions are satisfied:

(1) The topology τ of F coincides with $\tau_c(F, F')$.

(2) If a linear functional f on F maps bounded sets into bounded sets, then f is continuous.

PROOF. The "only if" part follows from [11, Theorems 4.30, 4.31]. Conversely, let the conditions be satisfied and let τ_b be the bornological topology associated with τ (τ_b is the finest locally K-convex topology on F having the same with τ bounded sets). If $f \in (F, \tau_b)'$, then f maps τ-bounded sets into bounded sets and so $f \in F'$. Since $\tau_b \geq \tau$, we have $(F, \tau_b)' = F'$ and so $\tau_b \leq \leq \tau_c(F, F') = \tau$. Thus $\tau = \tau_b$ is bornological.

PROPOSITION 2.14. Let K be spherically complete and let $G = C_s(X, E)$. Then, the topology of E coincides with $\tau_c(E, E')$ iff the topology of G coincides with $\tau_c(G, G')$.

PROOF. $(\Rightarrow)$ Let A be an absolutely convex, weakly c-compact and weakly bounded subset of G'. By Lemma 2.12, A is strongly bounded. By [8, Theorem 2.4], there exist a finite subset $S = \{x_1, \ldots, x_n\}$ of X and a strongly bounded subset H of E' such that

$$A \subset \{ \sum_{k=1}^{n} m_{x_k, \phi_k} : \phi_k \in H \}.$$

Consider pairwise disjoint clopen sets $W_1, \ldots, W_n$ in X with $x_k \in W_k$. For each k, the map $T_k : G'_\sigma \to E'_\sigma$, $T_k(m) = m(V_k)$, is linear and continuous. Hence, $T_k(A)$ is an absolutely convex, weakly bounded and weakly c-compact subset of E'. By hypothesis, $[T_k(A)]^0$ is a neighborhood of zero in E. Let p be a continuous non-Archimedean seminorm on E such that

$$V = \{ s \in E : p(s) \leq 1 \} \subset \bigcap_{k=1}^{n} [T_k(A)]^0.$$

Now

$$W = \{ f \in C(X, E) : \|f\|_{S, p} \leq 1 \} \subset A^0.$$

In fact, if $m \in A$, then for each $f \in C(X, E)$ we have

$$\int f dm = \sum_{k=1}^{n} \int_{W_k} f dm = \sum_{k=1}^{n} m(W_k) f(x_k).$$

Hence, for $m \in A$ and $f \in W$, we have $|m(f)| \leq 1$ and so $W \subset A^0$. It follows that the topology of G is finer that $\tau_c(G, G')$ and so the two topologies coincide.

$(\Leftarrow)$ Let $x \in X$. The map $T_x : E'_\sigma \to G'_\sigma$, $T_x(\phi) = m_{x, \phi}$, is linear and continuous. Hence, given an absolutely convex, weakly bounded and weakly c-compact subset B of E', the set $T_x(B)$ is an absolutely convex weakly bounded and weakly c-compact subset of G' and so its polar in G is a neighborhood of zero. Thus, there exist a finite subset S of X and a continuous non-Archimedean seminorm p on E such that

$$O = \{ f \in C(X, E) : \|f\|_{S, p} \leq 1 \} \subset [T_x(B)]^0.$$

Now, if $p(s) \leq 1$, then the constant function $f \equiv s$ is in O and so $1 \geq |T_x(\phi)(f)| = = |\phi(f(x))| = |\phi(s)|$ for all $\phi \in B$. Thus

$$\{ s \in E : p(s) \leq 1 \} \subset B^0$$

and the result follows.

THEOREM 2.15. Let K be spherically complete and suppose that X satisfies the first axiom of countability. Then $G=C_s(X,E)$ is bornological iff X is N-replete and E is bornological.

PROOF. The "only if" part follows from Theorem 2.1. Conversely, let E be bornological and X N-replete. Since E is bornological, its topology coincides with $\tau_c(E,E')$. By the preceding Proposition, $t_s=\tau_c(G,G')$. To finish the proof, it suffices (by Lemma 2.13) to show that every linear functional ϕ on G, bounded on bounded sets, is continuous. So, let ϕ be such a functional. The set $D=\{f\in C(X,E):|\phi(f)|\leq 1\}$ is absolutely convex absorbing t_s-bounded sets. Hence, $F=F(D)$ is a finite subset of $\upsilon_0 X=X$. If $f\in C(X,E)$ vanishes on F and $|\lambda|>1$, then $\lambda^n f\in D$ for all n (by Proposition 2.11) and so $|\phi(f)|\leq|\lambda|^{-n}$ which proves that $\phi(f)=0$. Let $F=\{x_1,\dots,x_n\}$. The space $E_1=E^n$, with the product topology, is bornological. Choose pairwise disjoint clopen sets $W_1,\dots,W_n$ in X with $x_k\in W_k$. For $s=(s_1,\dots,s_n)\in E_1$, define

$$T(s)=\phi\Big(\sum_{k=1}^{n}\chi_{W_k}s_k\Big).$$

Then T is linear and bounded on bounded sets in E_1, and hence T is continuous since E_1 is bornological. Let $\phi_k:E\to K$, $\phi_k(x)=T(x^{(k)})$, where

$$\begin{aligned} x_i^{(k)} &= 0 \text{ if } i\neq k\\ &= x \text{ if } i=k. \end{aligned}$$

Then, ϕ_k is continuous. Now, given $f\in C(X,E)$, the function

$$h=f-\sum_{k=1}^{n}\chi_{W_k}f(x_k)$$

vanishes on F and so $\phi(h)=0$, which implies that

$$\phi(f)=\phi\Big(\sum_{k=1}^{n}\chi_{W_k}f(x_k)\Big)=T(f(x_1),\dots,f(x_n))=\sum_{k=1}^{n}\phi_k(f(x_k)).$$

Thus $\phi\in G'$, which completes the proof.

3. c-ULTRABORNOLOGICAL SPACES $C(X,E)$

THEOREM 3.1. If $C_c(X,E)$ or $C_s(X,E)$ is c-ultrabornological, then E is c-ultrabornological, X is N-replete and K is spherically complete.

PROOF. Let $C_c(X,E)$ (resp. $C_s(X,E)$) be c-ultrabornological and let W be an absolutely convex subset of E absorbing absolutely convex, bounded and c-compact sets. If $x\in X$, then the set $D=\{f\in C(X,E):f(x)\in W\}$ is absolutely convex and absorbs absolutely convex, bounded and c-compact sets in $C_c(X,E)$ (resp. in $C_s(X,E)$). By hypothesis, there exist a compact subset F of X and a continuous non-Archimedean seminorm p on E such that

$$\{f\in C(X,E):\|f\|_{F,p}\leq 1\}\subset D.$$

This implies that the set $\{s \in E : p(s) \leq 1\} \subset W$ which shows that E is c-ultrabornological. Also, X is N-replete by Theorem 2.1. Finally, if there exists a c-ultrabornological Hausdorff space $G \neq \{0\}$ over K, then G contains an absolutely convex, bounded and c-compact set $A \neq \{0\}$ (otherwise $\{0\}$ would be a neighborhood of zero in G) which (since K is complete) it is known to imply that K is c-compact. This clearly completes the proof.

PROPOSITION 3.2. Let K be spherically complete and let E be a Fréchet space. If D is an absolutely convex subset of $C_c(X,E)$ absorbing absolutely convex, bounded and c-compact sets, then there exist a continuous non-Archimedean seminorm p on E and $r>0$ such that

$$\{f \in C(X,E) : \|f\|_p \leq r\} \subset D.$$

PROOF. Assume the contrary and let (p_n) be an increasing sequence of non-Archimedean seminorms on E generating its topology. If $|\lambda|>1$, there exists $f_n \in C(X,E)$ with $f_n \notin D$ and $\|f_n\|_{p_n} \leq |\lambda|^{-2n}$. Let (λ_m) be a sequence in K with $|\lambda_m| \leq 1$ for each m. Put $g_n = \sum_{k \leq n} \lambda_k \lambda^k f_k$. If $m > n \geq k$, then $\|g_m - g_n\|_{p_k} \leq$ $\leq |\lambda|^{-n}$. In fact, for $i > n$,

$$\|\lambda_i \lambda^i f_i\|_{p_k} \leq \|\lambda^i f_i\|_{p_i} \leq |\lambda|^{-i} \leq |\lambda|^{-n}.$$

Since E is complete, there exists $g \in C(X,E)$ such that $g_n \to g$ uniformly. Clearly $g = \sum_{n=1}^{\infty} \lambda_n \lambda^n f_n$. The set $V = \{\gamma \in K : |\gamma| \leq 1\}$ is absolutely convex and c-compact. Hence, if we consider on K^N the product topology, the set V^N is absolutely convex and c-compact in K^N by [10, 1.17]. The mapping

$$f : V^N \to C_c(X,E),\ (\lambda_m) \mapsto \sum_{n=1}^{\infty} \lambda_n \lambda^n f_n,$$

is continuous. Thus, the set $B = f(V^N)$ is absolutely convex, bounded and c-compact in $C_c(X,E)$. By hypothesis, there exists μ such that $B \subset \mu D$ and so $\lambda^n f_n \subset \mu D$ for each n. Hence $f_n \in D$ for sufficiently large n, which is a contradiction. This completes the proof.

PROPOSITION 3.3. Let K be spherically complete and let D be an absolutely convex subset of $C_c(X,E)$ absorbing absolutely convex, bounded and c-compact sets. Then $F(D) \subset \upsilon_0 X$.

PROOF. Suppose that $x \in F(D)$, $x \notin \upsilon_0 X$, and let (A_n), (G_n) be as in the proof of Proposition 2.4. Set $W_n = \bigcup_{k \leq n} G_k$. Since $\bar{G}_n^{\beta_0 X} = A_n \setminus A_{n+1}$, we have $x \notin \bar{W}_n^{\beta_0 X}$ and so $W_n \notin \Omega_D$. Let $f_n \in C(X,E)$ with $f_n \notin D$ and $f_n = 0$ on W_n. Let $|\lambda|>1$. For each sequence (λ_n) in K, the series $\sum_{n=1}^{\infty} \lambda_n \lambda^n f_n$ represents an element g of $C(X,E)$ (on W_n we have $g = \sum_{k \leq n-1} \lambda_k \lambda^k f_k$ and so g is continuous at each point of W_n). If $V = \{\gamma \in K : |\gamma| \leq 1\}$, then V^N is an absolutely convex and c-compact subset of K^N. The mapping

$$h : K^N \to C_c(X,E),\ (\lambda_n) \mapsto \sum_{n=1}^{\infty} \lambda_n \lambda^n f_n,$$

is continuous. In fact, if F is a compact subset of X, then $F \subset W_m$, for some m, and so $\sum_{n=1}^{\infty} \lambda_n \lambda^n f_n = \sum_{n \leq m} \lambda_n \lambda^n f_n$ on F. It follows that the set $B = h(V^N)$ is absolutely convex, bounded and c-compact in $C_c(X, E)$. Since B is absorbed by D, it follows that $f_n \in D$ eventually. This contradiction completes the proof.

Combining the preceding Propositions, we get

THEOREM 3.4. Let E be a Fréchet space. Then, $C_c(X, E)$ is c-ultrabornological iff X is N-replete and K is spherically complete.

PROOF. The "only iff" part follows from Theorem 3.1. Conversely, let X be N-replete and K spherically complete. Let D be an absolutely convex subset of $C_c(X, E)$ absorbing absolutely convex, bounded and c-compact sets. Then $F(D)$ is a compact subset of $\upsilon_0 X = X$ by Proposition 3.3. From Propositions 3.3 and 2.7, there exist a continuous non-Archimedean seminorm p on E and $r > 0$ such that

$$\{f \in C(X, E) : \|f\|_{F(D), p} \leq r\} \subset D$$

and so D is a neighborhood of zero in $C_c(X, E)$. This completes the proof.

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